Randomly modulated dark soliton



1991 J. Phys. A: Math. Gen. 24 767

(http://iopscience.iop.org/0305-4470/24/4/013)

J. Phys. A: Math. Gen. 24 (1991) 767–785. Printed in the UK

# Randomly modulated dark soliton

V V Konotop and V E Vekslerchik
Institute of Radiophysics and Electronics, UkrSSR Academy of Sciences, Academik Proskura street 12, Kharkov 310085, USSR

Received 18 June 1990

**Abstract.** The effect of initial fluctuations on the dynamics of dark solitons is studied in the framework of the nonlinear Schrödinger equation with non-zero boundary conditions. The perturbation method based on the inverse scattering technique is developed to obtain statistical characteristics of soliton parameters. The possibility of creation of new solitons due to perturbation is considered and a criterion for that to occur is stated. The features of the soliton dynamics in comparison with those of bright solitons are discussed.

## 1. Introduction

In a series of recent experiments (Krökel *et al.* 1988, Weiner *et al* 1988; see also Tomlinson 1988) carried out by different groups, dark optical pulses propagating in single-mode fibres with positive group velocity dispersion have been observed. Such pulses are the intensity dips of a background wave. In the experiments the dips were of the order of 0.3 ps (Krökel *et al* 1988) and 0.185 ps (Weiner *et al* 1988) while the background pulses were of the order of 100 and 4 ps correspondingly. It is well known (Hasegawa and Tappert 1973) that optical pulses of the above-mentioned durations are described by the nonlinear Schrödinger equation (NSE) or, more precisely, by the 'stable' NSE (Zakharov and Shabat 1973). It allowed the existence of dark solitons (as the stable dark pulses have been called) to be predicted as long ago as 1973 (Hasegawa and Tappert 1973, Zakharov and Shabat 1973). The experimental progress stimulated theoretical investigations devoted to optical applications of the stable NSE (Gredeskul *et al* 1989, Kivshar 1990). The absence of a threshold for generation of dark solitons, experimentally observed by Krökel *et al* (1989), as well as criterion for the existence of dark solitons have been stated theoretically by Gredeskul and Kivshar (1989). Tomlinson and co-workers (1989a, b) have investigated numerically the influence of the finite width of background pulses on dark soliton stability and have stated that the dark pulse maintains its soliton characteristics as the background pulse evolves. Gredeskul *et al* (1989) have considered analytically evolution of pulses with various initial shapes including dark pulses on a finite width background (in the WKB approximation) and random ones in the non-soliton case. In general, fluctuations are the attribute of all experiments. In particular, input fluctuations may lead to restrictions on the length of an optical fibre used for investigations of soliton properties (Krökel *et al* 1988).

Analogous problems arise in connection with bright soliton dynamics (i.e. at laser-beam wavelength from the region of the negative group velocity dispersion) and have been studied in a number of papers (Elgin 1985, Vysloukh *et al* 1987, Bass *et al*

1987, 1988, Konotop 1989). Briefly, the behaviour of a randomly modulated bright soliton is as follows. Initial amplitude and phase modulations change its parameters. If fluctuations are weak enough, the amplitude and velocity are distributed according to the Gaussian laws. Also, the random modulation raises optical noise in a fibre. Initially this noise is located in the soliton region. But during its propagation the soliton 'clears' itself from the noise. Statistical characteristics of the noise are similar to those in the linear case (i.e. without solitons).

The dynamics of a randomly modulated dark soliton have been numerically studied by Zhao and Bourkoff (1989). They stated the conditions needed for the initial pulse to decay into the prime and two small solitons and discussed the effect of both fluctuations and losses on the soliton dynamics. In particular, it was found that dark solitons are less affected by dissipation and background noise than bright ones.

In the present paper we consider analytically the effect of initial random modulations on dark soliton dynamics. All calculations are carried out within the framework of the NSE with non-zero boundary conditions at infinity, which corresponds to an endless background pulse. The mathematical statement of the problem as well as a very brief review of some aspects of the inverse scattering technique (IST) are presented in section 2. In section 3 we develop the perturbation theory for a modulated dark soliton based on the IST. In section 4 we obtain statistical characteristics of soliton parameters for initial fluctuations of rather general form. There we also consider two limiting cases of small and large correlation radii of initial fluctuations and the case of a random phase modulation. The possibility of the creation of new solitons due to initial fluctuations is discussed in section 5, where we consider some particular cases, and in section 6, where a general approach to the problem is presented. The conclusion is devoted to discussion of the results obtained.

## 2. Mathematical remarks

Optical pulse propagation in a single-mode optical fibre in the region of normal group velocity dispersion is described by the NSE (Hasegawa and Tappert 1973; see also Tomlinson *et al* 1989), presented in the traditional form as

$$\mathrm{i}q_t + q_{xx} - 2|q|^2 q = 0. \tag{1}$$

Here $q$ is the complex field envelope, normalized to make $|q|^2 = 1$ correspond to an intensity of pulse equal to $10^{-7} nc\lambda_0 / 16\pi n_2 z_0$ W cm$^{-2}$. The variable $t$ is the coordinate along the fibre normalized onto $4z_0/\pi$, while $x$ is the retarded time in the frame of the carrier mode, normalized to the input pulse duration $t_0$; $z_0 = (\pi c t_0)^2 / \lambda_0 |\mathrm{D}(\lambda_0)|$, $\mathrm{D}(\lambda_0) = \lambda_0^2 \mathrm{d}^2 n / \mathrm{d}\lambda_0^2$, $\lambda_0$ is the carrier mode wavelength; $n$ is the refractive index and $n_2$ is Kerr's coefficient.

The substitution $q \to q \exp(-2\mathrm{i}p^2 t)$ modifies (1) to

$$\mathrm{i}q_t + q_{xx} + 2(p^2 - |q|^2)q = 0 \tag{2}$$

under the boundary conditions

$$\lim_{x \to -\infty} q = p \qquad \lim_{x \to +\infty} q = p \exp(\mathrm{i}\theta). \tag{3}$$

Here $p$ is a constant taken as positive real and designates the background pulse amplitude, and $\theta$ is a complete phaseshift.

It is well known (Zakharov and Shabat 1973) that the Cauchy problem for the NSE may be solved exactly by means of the IST. The IST is well covered in numerous works, and thus we outline here, following Takhtajan and Faddeev (1986), only the principal points of this method that will be used below.

One can associate a linear scattering problem, the so-called Zakharov-Shabat system, with the NSE (no matter which of its forms we consider: equation (1) or (2)) such that the initial condition $q(x) = q(x, t=0)$ forms the scattering potential

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x; \lambda) = U(x; \lambda) F(x; \lambda). \tag{4}$$

Here $F$ is a $2 \times 2$ matrix,

$$U(x; \lambda) = \begin{pmatrix} \dfrac{\lambda}{2\mathrm{i}} & \bar{q}(x) \\ q(x) & -\dfrac{\lambda}{2\mathrm{i}} \end{pmatrix} \tag{5}$$

$\lambda$ is a spectral parameter and the overbar denotes complex conjugation.

By the known result (Zakharov and Shabat 1973, Takhtajan and Faddeev 1986) the spectral problem (4) has the continuous spectrum $R_\omega$ at real $\lambda$: $\lambda^2 > \omega^2$, where $\omega$ is defined as $\omega = 2p$. Also, (4) may have eigenvalues in the lacuna $(-\omega, \omega)$ constituting the discrete spectrum.

The so-called scattering data, which play an important role in the IST, can be obtained from the scattering matrix $T(\lambda)$ linking the normalized solutions of (4):

$$T_-(x; \lambda) = T_+(x; \lambda) T(\lambda). \tag{6}$$

Matrix Jost functions $T_\pm(x; \lambda)$ are defined by their asymptotics:

$$T_\pm(x; \lambda) \sim \exp\left(\frac{1 \pm 1}{4\mathrm{i}} \theta \sigma_3\right) E(x; \lambda) \qquad \text{as } x \to \pm\infty. \tag{7}$$

Here

$$E(x; \lambda) = \begin{pmatrix} 1 & \dfrac{\omega}{\mathrm{i} z(\lambda)} \\ \dfrac{\mathrm{i}\omega}{z(\lambda)} & 1 \end{pmatrix} \exp\left[\frac{k(\lambda) x}{2\mathrm{i}}\right] \tag{8}$$

$\sigma_3 = \mathrm{diag}(1, -1)$ is a Pauli matrix, and parameters $z$ and $k$ are introduced according to the relations

$$k(\lambda) = \sqrt{\lambda^2 - \omega^2} \qquad \operatorname{sgn} k = \operatorname{sgn} \lambda \qquad (\lambda \in R_\omega) \tag{9}$$

$$z(\lambda) = \lambda + k(\lambda). \tag{10}$$

For $\lambda \in R_\omega$ the scattering matrix $T(\lambda)$ may be written in the form

$$T(\lambda) = \begin{pmatrix} a(\lambda) & \bar{b}(\lambda) \\ b(\lambda) & \bar{a}(\lambda) \end{pmatrix}. \tag{11}$$

If we consider $a(\lambda)$ as a function of $z$, defined by (10), the set of zeros of $a(\lambda(z))$ in the upper half-plane of complex $z$,

$$z_n = \lambda_n + \mathrm{i}\nu_n \qquad (n = 1, \ldots, N) \tag{12}$$

where

$$\nu_n = \sqrt{\omega^2 - \lambda_n^2} > 0 \tag{13}$$

and $N$ is the number of eigenvalues of the scattering problem, is in the one-to-one correspondence with the discrete spectrum of the scattering problem.

Since for all $n$

$$|z_n| = \omega \tag{14}$$

the discrete spectrum may be characterized by the set of angles $\{\theta_n\}$ defined by

$$z_n = -\omega \exp\left(-\frac{i}{2}\theta_n\right) \tag{15}$$

(see figure 1 and below).

It follows from the analysis of analytical properties of the scattering matrix that

$$a(\lambda) = \exp\left(\frac{i}{2}\theta\right) \prod_{n=1}^{N} \frac{z(\lambda) - z_n}{z(\lambda) - \bar{z}_n} \exp\left[\frac{1}{2\pi i}\int_{R_\omega} \frac{d\mu}{k(\mu)}\left(1 + \frac{k(\lambda)}{\mu - \lambda}\right)\ln|a(\mu)|^2\right]. \tag{16}$$

This representation together with the asymptotics of $a(\lambda)$ for large $\lambda$,

$$a(\lambda) = \exp\left(\pm\frac{i}{2}\theta\right) + O(|\lambda|^{-1}) \tag{17}$$

when $\pm\mathrm{Im}\,\lambda > 0$, gives a relation which will be used below (the so-called $\theta$-relation, Takhtajan and Faddeev 1986)

$$\exp(-i\theta) = \prod_{n=1}^{N} \exp(-i\theta_n) \exp\left(\frac{1}{\pi i}\int_{R_\omega} \frac{d\lambda}{k(\lambda)} \ln|a(\lambda)|^2\right). \tag{18}$$

The NSE as a completely integrable system possesses an infinite number of the conservation laws. Further, we shall make use of one of them which may be written in the form (Takhtajan and Faddeev 1986)

$$\tfrac{1}{2}\sum_n \lambda_n \nu_n = \frac{1}{2\pi}\int_{R_\omega} \frac{d\lambda}{k(\lambda)} (\lambda^2 - \tfrac{1}{2}\omega^2) \ln|a(\lambda)|^2 - J_2[q] \tag{19}$$

where

$$J_2[q] = \frac{1}{2i}\int_{-\infty}^{\infty} dx(q_x\bar{q} - q\bar{q}_x). \tag{20}$$

The identity (19) may be considered as the law of conservation of momentum.

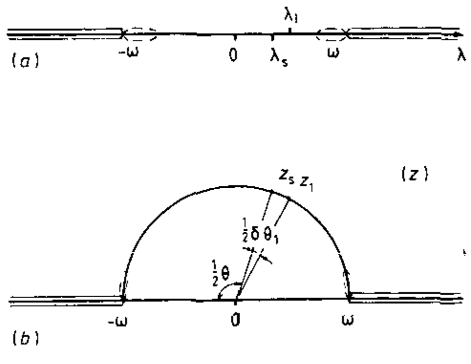


**Figure 1.** Spectrum of the scattering problem, soliton parameters in terms of $\lambda(a)$ and $z(b)$. Regions near the points $\pm\omega$ are those where appearance of new solitons can occur.

## 3. Perturbation theory

The complete analytical solution of the Zakharov–Shabat system (4) may be done only for some particular cases. The most interesting of them is the refractionless, or multi-soliton, case: $b(\lambda)=0$. The one-soliton potential ($N=1$) has the form

$$q_s(x)=p\frac{1+\varepsilon^2\exp(\nu x)}{1+\exp(\nu x)}. \tag{21}$$

Here the subscript for $\nu$ is omitted and the notation $\varepsilon=\exp(\mathrm{i}\theta/2)$ is introduced. The scattering matrix for this potential is $T_s(\lambda)=\mathrm{diag}(a_s(\lambda), a_s(\lambda)^{-1})$. The Jost coefficient

$$a_s(\lambda)=\varepsilon\frac{z+\omega\bar{\varepsilon}}{z+\omega\varepsilon} \qquad z=z(\lambda) \tag{22}$$

considered as a function of $z$, has one zero

$$z_s=\lambda_s+\mathrm{i}\nu=-\omega\bar{\varepsilon}. \tag{23}$$

The one-soliton Jost functions are presented in appendix A.

In the generic case it is impossible to solve (4) exactly. But if the difference

$$\delta q(x)=q(x, t=0)-q_s(x) \tag{24}$$

is sufficiently small, the perturbation theory may be developed. For the unstable NSE, such theories have been constructed by Elgin and Kaup (1982), Elgin (1985), Vysloukh *et al* (1987), Bass *et al* (1987, 1988) and Konotop (1989).

By analogy with the case of bright soliton dynamics, it is naturally expected that initial perturbations result in the changing of soliton parameters and the generation of quasilinear modes. Also, a new soliton or a pair of new solitons can be created (a fact that was not observed within the framework of the perturbation theory for bright soliton dynamics). The question of what occurs when the soliton number changes will be discussed below.

The parameters of one of the solitons generated from the initial random pulse (it will be called 'prime') are close to those of the unperturbed soliton. Its velocity $\lambda_1$ and deep $\nu_1$ of the intensity hole relative to the background intensity may be written as follows

$$\lambda_1=-\omega\cos\frac{\theta_1}{2} \tag{25}$$

$$\nu_1=\omega\sin\frac{\theta_1}{2} \tag{26}$$

with $\theta_1\approx\theta$ (cf. (23)). The main subject of this section is the variation of the parameters of the prime soliton, $\delta\lambda_1=\lambda_1-\lambda_s$ and $\delta\nu_1=\nu_1-\nu_s$, which may be expressed in terms of $\delta\theta_1=\theta_1-\theta$:

$$\delta\lambda_1=\tfrac{1}{2}\nu\delta\theta_1-\tfrac{1}{8}\lambda_s(\delta\theta_1)^2+\mathrm{O}((\delta\theta_1)^3) \tag{27}$$

$$\nu_1=-\tfrac{1}{2}\lambda_s\delta\theta_1-\tfrac{1}{8}\nu(\delta\theta_1)^2+\mathrm{O}((\delta\theta_1)^3). \tag{28}$$

From the mathematical point of view, in situations when the perturbation method is applicable, the creation of new solitons is possible only near the edges of the continuous spectrum. Hence, the 'newborn' solitons, in case they exist, have small amplitudes and large velocities: $\nu_n \approx 0$ and $\lambda_n \approx \pm\omega$ for $n > 1$. Their parameters may be written in a form similar to (25) and (26) with

$$\theta_n = \begin{cases} \delta\theta_n & \text{if } \lambda_n \approx -\omega \\ 2\pi - \delta\theta_n & \text{if } \lambda_n \approx \omega \end{cases} \qquad (n > 1) \tag{29}$$

where $\delta\theta_n$ for $n > 1$, as well as $\delta\theta_1$, are of the order of the initial perturbation $\delta q$.

A small variation $\delta U$ of the matrix $U$ caused by $\delta q$ in (4) causes a variation of the scattering matrix, which to the first order in $\delta U$ may be written as follows (Takhtajan and Faddeev 1986):

$$\delta T(\lambda) = \int_{-\infty}^{\infty} T_+^{-1}(x;\lambda)\,\delta U(x)\,T_-(x;\lambda)\,\mathrm{d}x \tag{30}$$

where $T_\pm$ are unperturbed Jost functions, i.e. the Jost functions of the system (4) and (5) with $q = q_s$ (see (A1)).

Since

$$\det T_\pm(x;\lambda) = \frac{2k(\lambda)}{z(\lambda)} \tag{31}$$

matrices $T_+^{-1}$ and, hence, $\delta T$ have singularities at $\lambda = \pm\omega$. This is an exhibition of the fact that in the case of non-zero boundary conditions the points $\lambda = \pm\omega$ are, in the generic case, the singular points of the scattering matrix:

$$a|_{\lambda \approx \pm\omega} = \frac{a_\pm}{k(\lambda)} + \mathrm{O}(1) \tag{32}$$

$$b|_{\lambda \approx \pm\omega} = \frac{b_\pm}{k(\lambda)} + \mathrm{O}(1). \tag{33}$$

One can show (e.g. see Takhtajan and Faddeev 1986) that 'residues' $a_\pm$ and $b_\pm$ (which are residues indeed if we consider the scattering matrix as a function of $z$) are related by the formula

$$a_\pm = \pm\mathrm{i}b_\pm \tag{34}$$

and that $b_\pm$ are real.

Thus we cannot consider $\delta T$ as small when $\delta U$ is small (as it was in the case of a perturbed bright soliton) in spite of the fact that $\delta T = 0$ when $\delta U = 0$.

To take into account the above-mentioned singularities of the Jost coefficient $a(\lambda)$ consider the function $C(\lambda)$ defined by

$$a(\lambda) = a_s(\lambda)\left(1 + \mathrm{i}\frac{C(\lambda)}{k(\lambda)}\right). \tag{35}$$

The quantity $C(\lambda)$ is regular for all $\lambda$ from $R_\omega$, and is small as far as $\delta q$ is small. One can show that to the lowest order in fluctuations $C(\lambda)$ is real for real $\lambda$. It is presented in appendix B.

Since the relations between $\lambda_n$, $\nu_n$ and $\theta_n$ $(n > 1)$ are similar to (25) and (26), one can conclude that

$$\tfrac{1}{2}\omega^2\delta\theta_n - |\lambda_n|\nu_n = \mathrm{O}((\delta q)^3) \qquad (n > 1). \tag{36}$$

Then, by using (18) and (19) we obtain to the second order in $\delta q$:

$$\delta\theta_1 - [\sin(\theta + \delta\theta_1) - \sin\theta] = \frac{2}{\pi\omega^2}\int_{R_\omega} \mathrm{d}\lambda\, k(\lambda)\ln|a(\lambda)|^2 - \frac{4}{\omega^2}\delta J_2 \tag{37}$$

where $\delta J_2 = J_2[q_s+\delta q] - J_2[q_s]$ and $J_2(q)$ is defined by (20). It can be shown that

$$\int_{R_\omega} \mathrm{d}\lambda\, k(\lambda)\ln|a(\lambda)|^2 = \int_{R_\omega} \frac{\mathrm{d}\lambda}{k(\lambda)} C^2(\lambda) + \mathrm{O}((\delta q)^3) \tag{38}$$

where $C(\lambda)$ is given by (35). A derivation of this relation is outlined in appendix C. Now, by iterating (37), one can get, working to order $(\delta q)^2$, the following expression for $\delta\theta_1$:

$$\delta\theta_1 = \frac{2}{\nu^2}\left(-\delta J_2 + \frac{\lambda_s}{\nu^3}(\delta J_2)^2 + \frac{1}{2\pi}\int_{R_\omega} \frac{\mathrm{d}\lambda}{k(\lambda)} C^2(\lambda)\right). \tag{39}$$

## 4. Statistical characteristics of dark soliton parameters

In this section we obtain the statistical characteristics of the prime soliton parameters for two classes of initial perturbations which are usually used to describe fluctuations of real laser pulses.

First consider the case of phase modulations:

$$q(x, t=0) = q_s(x)\exp[\mathrm{i}\varphi(x)]. \tag{40}$$

Here $\varphi(x)$ is a real random process of zero mean value and correlator

$$\langle\varphi(x)\varphi(x')\rangle = \Phi(x-x') = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{\Phi}(\xi)\exp\left[\mathrm{i}\frac{\xi}{2}(x-x')\right]. \tag{41}$$

Inserting (41) in both the definition of $J_2$ and expression for $C(\lambda)$ (B2) one can get

$$\begin{pmatrix}\delta J_2\\ C(\lambda)\end{pmatrix} = \begin{pmatrix}1\\ (\lambda-\lambda_s)^{-1}\end{pmatrix}\int_{-\infty}^{\infty} \mathrm{d}x(|q_s|^2 - p^2)\varphi_x. \tag{42}$$

Then, simple algebra invoking the direct averaging of (39) and evaluating the integral over $R_\omega$ in (39) yields the mean value of $\delta\theta_1$:

$$\langle\delta\theta_1\rangle_{\mathrm{ph}} = 3\frac{\lambda_s}{\nu}\Delta^2_{\mathrm{ph}}. \tag{43}$$

Together with (39) it also gives

$$\langle(\delta\theta_1)^2\rangle_{\mathrm{ph}} = 4\Delta^2_{\mathrm{ph}} \tag{44}$$

where

$$\Delta^2_{\mathrm{ph}} = \left(\frac{\pi}{4\nu^2}\right)^2 \int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{\Phi}(\xi)\xi^4 \operatorname{cosech}^2\frac{\pi\xi}{2\nu}. \tag{45}$$

The subscript 'ph' is added to indicate phase fluctuation.

In the case of sufficiently weak Gaussian initial fluctuations the random quantity $\delta\theta_1$, and therefore $\delta\lambda_1$ and $\delta\nu_1$ (see (27) and (28)), may be regarded as distributed according to the normal law. Then formulae (43)–(45) completely describe the statistics of $\delta\theta_1$, and hence, together with (27) and (28), statistics of the additional parameters $\delta\lambda_1$ and $\delta\nu_1$. So, the mean values of the shifts of velocity and amplitude due to phase modulation (40) may be written as

$$\langle\delta\lambda_1\rangle_{\mathrm{ph}} = \lambda_s\Delta^2_{\mathrm{ph}} \tag{46}$$

$$\langle\delta\nu_1\rangle_{\mathrm{ph}} = -\nu\left[\frac{3}{2}\left(\frac{\lambda_s}{\nu}\right)^2 + \frac{1}{2}\right]\Delta^2_{\mathrm{ph}}. \tag{47}$$

Let us obtain the statistical characteristics of a dark soliton generated from an initial random pulse of the form $q(x, t=0) = q_s(x) + \delta q(x)$ with

$$\delta q(x) = g(x)\psi(x). \tag{48}$$

Here $g(x)$ is a regular function vanishing at infinity and $\psi(x)$ is a random process with zero mean value and correlators

$$\langle \psi(x)\psi(x')\rangle = A(x-x') = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{A}(\xi) \exp\left(\mathrm{i}\frac{\xi}{2}(x-x')\right) \tag{49}$$

$$\langle \psi(x)\bar{\psi}(x')\rangle = B(x-x') = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{B}(\xi) \exp\left(\mathrm{i}\frac{\xi}{2}(x-x')\right). \tag{50}$$

Note that such a choice of $\delta q$ is natural. Thus, in the experiments by Krökel *et al* (1988) the 'ultrafast light-controlled optical fibre modulator' has been used. The work of this device is based on the interaction of a long signal pulse with a driving one (Halas *et al* 1987). Then the additional parameter $\delta q$ may be considered as a random modulation of the driving pulse.

Below, for the sake of simplicity, we use the traditional form of a laser signal envelope:

$$g(x) = \operatorname{sech}\frac{\nu x}{2}. \tag{51}$$

For $\delta q$ given by (48) and (51), uncomplicated but rather cumbersome calculations give

$$\langle \delta J_2\rangle = \frac{2}{\nu}\int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{B}(\xi)\xi \tag{52}$$

$$\langle (\delta J_2)^2\rangle = \frac{\pi^2}{2\nu^2}\int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{D}(\xi)(\xi^2+\nu^2)^2 \operatorname{sech}^2\frac{\pi\xi}{2\nu} \tag{53}$$

$$\frac{1}{2\pi}\int_{R_\omega} \frac{\mathrm{d}\lambda}{k(\lambda)}\langle C^2(\lambda)\rangle = \int_{-\infty}^{\infty} \mathrm{d}\xi\, (\hat{B}(\xi)\xi Y(\xi) + \hat{D}(\xi)Z(\xi)) \tag{54}$$

where

$$\hat{D}(\xi) = \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\theta}\hat{A}(\xi) + \hat{B}(\xi)) \tag{55}$$

$$Y(\xi) = \frac{\pi^2}{2\nu^3}(\xi^2+\nu^2)\operatorname{sech}^2\frac{\pi\xi}{2\nu} \tag{56}$$

$$Z(\xi) = \frac{\pi^2\lambda_s}{16\nu^5}(\xi^2+\nu^2)(\xi^2-7\nu^2)\operatorname{sech}^2\frac{\pi\xi}{2\nu}. \tag{57}$$

Formulae (52)–(57) may be combined to give $\langle \delta\theta_1\rangle$ and $\langle(\delta\theta_1)^2\rangle$. To make the results obtained apparent we shall deal further with two limiting cases where the general formulae may be essentially simplified.

If a scale of fluctuations $R$ (or, in other words, a correlation radius) is much less than the dark soliton width, $R \ll \nu^{-1}$, it is natural to make a use of the delta-correlated random process approximation. Correspondingly, assume

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \begin{pmatrix} \exp(2\mathrm{i}\alpha + \mathrm{i}\theta) \\ 1 \end{pmatrix} \sigma_0^2 \delta(x). \tag{58}$$

Then, formulae (52)-(57) together with (27) and (28) give

$$\langle\delta\lambda_1\rangle_0 = -\tfrac{3}{2}\lambda_s\Delta_0^2 \tag{59}$$

$$\langle\delta\nu_1\rangle_0 = \nu\left[\left(\frac{\lambda_s}{\nu}\right)^2 - \frac{1}{2}\right] \tag{60}$$

where

$$\Delta_0^2 = \frac{8}{15\nu}(\sigma\cos\alpha)^2. \tag{61}$$

Similar results may be obtained in the opposite limiting case,

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \begin{pmatrix} \exp(2i\alpha + i\theta) \\ 1 \end{pmatrix}\sigma_\infty^2 \tag{62}$$

which corresponds to fluctuations with large correlation radius ($R \gg \nu^{-1}$). In this case

$$\langle\delta\lambda_1\rangle_\infty = -3\lambda_s\Delta_\infty^2 \tag{63}$$

$$\langle\delta\nu_1\rangle_\infty = \nu\left[\frac{5}{2}\left(\frac{\lambda_s}{\nu}\right)^2 - \frac{1}{2}\right]\Delta_\infty^2 \tag{64}$$

with

$$\Delta_\infty^2 = \left(\frac{\pi\sigma_\infty\cos\alpha}{2\nu}\right)^2. \tag{65}$$

As for the dispersion of the soliton parameters, the corresponding formulae for all the cases considered above (initial phase fluctuations (40); delta-correlated fluctuations (58) and large-scale initial amplitude modulations (62)) can be unified as follows:

$$\langle(\delta\lambda_1)^2\rangle = \nu^2\Delta^2 \tag{66}$$

$$\langle(\delta\nu_1)^2\rangle = \lambda_s^2\Delta^2 \tag{67}$$

where $\Delta$ is either $\Delta_{ph}$, $\Delta_0$ or $\Delta_\infty$.

Let us discuss the physical consequences of the results obtained. Since all $\Delta^2$ are positive, formulae (46), (59) and (63) imply that the average velocity of a dark soliton increases under initial phase modulations and decreases in the case of amplitude perturbations. Mean velocity is not changed if its unperturbed value is equal to zero, i.e. a rest soliton remains at rest in all the cases considered above. The dispersion of velocity fluctuations (see (66)) rises under the modulations of both types. The dependence of the average amplitude of a dark soliton on the initial conditions is more complicated. If phase fluctuations result in the decreasing of the mean deepness against a background carrier wave, the effect of amplitude fluctuations depends on the correlation radius and parameters of the unperturbed soliton. It can be seen from (60) and (64) that there is a critical velocity $\lambda_c$ which depends on the correlation radius $R$, such that initial amplitude fluctuations do not change (on average) the soliton amplitude if it corresponds to the eigenvalues $\lambda_s = \pm\lambda_c$: $\langle\delta\nu_1\rangle|_{\lambda_s=\pm\lambda_c} = 0$. So, for delta-correlated fluctuations $\lambda_c(R=0) = \omega/\sqrt{3}$ and for large correlation radii $\lambda_c(R=\infty) = \omega/\sqrt{6}$. Then, from (60) and (64), a slow soliton ($|\lambda_s| < \lambda_c$) becomes more shallow, while a fast one ($|\lambda_s| > \lambda_c$) becomes more deep.

In the case of a phase-modulated soliton the value of the correlation radius manifests itself only quantitatively, in the strength of fluctuations of soliton parameters. Amplitudes of the additional parameters $\langle\delta\lambda_1\rangle$ and $\langle\delta\nu_1\rangle$ as well as dispersions of the soliton parameter fluctuations reach their maxima as $R\to 0$ and vanish as $R\to\infty$ (it is assumed that the correlation function is correspondingly normalized).

The effect of initial amplitude perturbations essentially depends on the phase $\alpha$ as well. In particular, fluctuations and shifts of the prime soliton parameters decrease to zero with $\alpha$ changing from 0 to $\pm\pi/2$. If $\alpha=\pi/2$, both soliton and noise are phase contrast; when $\alpha=0$ they are phase coherent.

In conclusion of this section we consider an important dependence of our results on the unperturbed soliton amplitude $\nu$. If $\nu\to 0$ ($\lambda_s\to\pm\omega$) all mean values go to infinity in the case of amplitude noise (48) and to a non-zero value in the case of phase fluctuations. Mathematically it is stipulated by the singularities at the edges of the continuous spectrum (they have been discussed in the previous sections). It means that the developed perturbation theory fails near the points $\pm\omega$ (which corresponds to 'relativistic' dark solitons). To specify this region and, hence, to give estimates for the perturbation theory to be valid, we require

$$\Delta<\text{constant}\,\frac{\nu}{\lambda_s} \tag{68}$$

in the limit $\nu\to 0$ ($\lambda_s\to\pm\omega$). This condition provides (in the statistical meaning) the location of the perturbed root $\lambda_1$ inside the interval $(-\omega,\omega)$. The constant in (68) is of order one and has to be determined in each case separately. If the requirement (68) fails, the results for mean values are non-physical. In any case, small and fast dark solitons are strongly affected by the initial noise.

## 5. On the creation of new solitons: particular examples

The problems of the generation and disappearance of the bright solitons resulting from the noise have been discussed in the literature (e.g. see Kandidov and Schlyonov 1984, Elyutin *et al* 1988, Konotop 1989). However, considering the dynamics of an initially modulated bright soliton one can neglect the possibility of a new soliton arising (as well as the initial one being destroyed). A simple explanation of this fact may be given as follows. It is apparent that a 'newborn' soliton is of small amplitude if the initial perturbation is small enough. This implies that the corresponding zero of the Jost coefficient $\lambda_{\rm new}(a(\lambda_{\rm new})=0)$ is located near the real axis (the amplitude of a bright soliton is proportional to the imaginary part of the corresponding eigenvalue of the scattering problem). On the other hand the Jost coefficient $a(\lambda)$ in the case under consideration may be written as $a(\lambda)=a_s(\lambda)+\delta a(\lambda)$, where $a_s$ is the 'one-bright-soliton' Jost coefficient and an additional parameter $\delta a$ is due to initial fluctuations. For real $\lambda$ $|a_s(\lambda)|=1$, hence near the real axis $|a_s(\lambda)|\approx 1$. That is why $|\delta a(\lambda_{\rm new})|$ has to be close to unity. As is known, the probability for small fluctuations to make $\delta a$ to be of order one is exponentially small. Hence the possibility of creation of a new soliton may be neglected as long as the amplitude of the initial one is much more than the fluctuations.

The situation is different in the dark soliton case. Representation (35) shows that in spite of the smallness of $C(\lambda)$, which is of the order of fluctuations, the additional parameter $\delta a$ takes all values because of the factor $k(\lambda)^{-1}$. Thus we have to take into account the possibility of new eigenvalues appearing near the singular points $\pm\omega$. Note that eigenvalues close to $\pm\omega$ correspond to solitons of small amplitudes. In this region a pair of solitons is born from a pure stochastic pulse against a non-soliton background (Gredeskul and Kivshar 1989).

We cannot give an exhaustive answer to the question of how many dark solitons will be created due to an initial perturbation $\delta q$. But analysis presented in this section provides some insight into the problem.

First, consider the case when the unperturbed soliton is the rest one:

$$q_{s0}(x) = -\tfrac{1}{2}\omega \tanh(\tfrac{1}{2}\omega x) \tag{69}$$

i.e. the dark pulse given by (21) with $\theta$ being taken equal to $\pi$ (such a soliton is also called a black soliton in contrast to a grey one at $\theta \neq \pi$). Using the presentation (35) together with (B2) and (B3) the Jost coefficient $a(\lambda)$ may be written to first order in $\delta q$ as

$$a(\lambda) = a_s(\lambda)\left[1 + \frac{2i}{k(\lambda)}\left(\frac{\omega}{\lambda}\Delta_i + \Delta_r\right)\right] \tag{70}$$

where

$$\begin{pmatrix}\Delta_i \\ \Delta_r\end{pmatrix} = \int_{-\infty}^{\infty} dx\, q_{s0}(x) \begin{pmatrix}\dfrac{1}{\omega}\dfrac{d}{dx}\,\mathrm{Im}\,\delta q(x) \\ \mathrm{Re}\,\delta q(x)\end{pmatrix}. \tag{71}$$

Note that $\lambda = 0$ ($a_s(0) = 0$) is no longer a root of the dispersion relation $a(\lambda) = 0$, since the second term in brackets in the RHS of (70) diverges as $\lambda \to 0$. Equating $a(\lambda)$ with zero one can obtain the equation for the discrete spectrum of the scattering problem

$$\lambda\sqrt{\omega^2 - \lambda^2} + 2(\omega\Delta_i + \lambda\Delta_r) = 0. \tag{72}$$

It can be shown that this equation has an even number of roots if $|\Delta_i| > |\Delta_r|$ and an odd number if $|\Delta_i| < |\Delta_r|$. Now we illustrate this fact by two examples.

As the first example consider the creation of an additional soliton caused by a weak initial phase modulation of the black soliton, i.e. the case when

$$q(x, t=0) = q_{s0}(x)\exp(i\varphi(x)) \tag{73}$$

where $\varphi$ is small, and $\varphi(\pm\infty) = 0$. By solving (72) with $\Delta_r = 0$ and

$$\Delta_i = \frac{1}{2\omega}\int_{-\infty}^{\infty} dx\, q_{s0}^2(x)\varphi'(x) \tag{74}$$

(the prime denotes differentiation with respect to $x$) it is easy to show that there are two eigenvalues: $\lambda_1 = -2\Delta_i$, $\lambda_2 = -(\omega - 2\omega^{-1}\Delta_i^2)\,\mathrm{sgn}\,\Delta_i$. In other words, two solitons will be created from the initial packet given by (73): the prime one with 'amplitude' $\nu_1 = \omega - 2\omega^{-1}\Delta_i^2$ and a more fast additional one with $\nu_2 = 2|\Delta_i|$ (see figure 2). Both solitons move in the same direction.

This situation differs essentially from the bright soliton case, when phase modulations lead to the decreasing of the soliton amplitude only (Konotop 1989), and is a manifestation of the 'phase nature' of dark solitons.

As another example consider the case of antisymmetric perturbations,

$$\delta q(x) = -\delta q(-x). \tag{75}$$

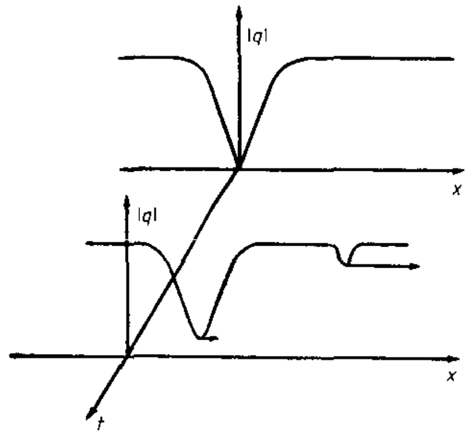


**Figure 2.** Schematic evolution of a perturbed black soliton in the case of an initial phase modulation.

In this case $\Delta_i = 0$, and it is clear that (72) has a solution $\lambda_1 = 0$ (antisymmetric fluctuations have no effect on the black soliton). As for the additional solitons, there will be a pair of them if $\Delta_r < 0$. They correspond to the pair of symmetrically situated eigenvalues

$$\lambda_2 = -\lambda_3 = \omega - 2\omega^{-1}\Delta_r^2 \tag{76}$$

(cf. results obtained by Gredeskul and Kivshar (1989)). Being of small amplitudes $\nu_2 = \nu_3 = 2|\Delta_r|$ these solitons will propagate with equal (in modulus) velocities in opposite directions (see figure 3).

As was mentioned in the introduction, Zhao and Bourkoff (1989) have considered the evolution of the packet, which in our notation may be written in the form

$$q(x, t=0) = -\tfrac{1}{2}\omega \tanh(\tfrac{1}{2}\Omega x). \tag{77}$$

When $\Omega$ is close to $\omega$ this case may be treated as the case of an antisymmetrically perturbed dark soliton. Our criterion for the creation of a new pair of solitons ($\Delta_r < 0$),

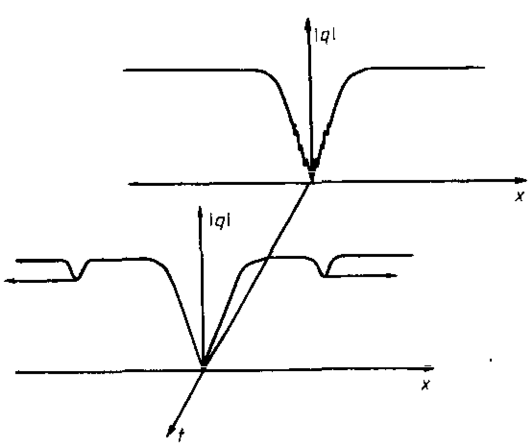


**Figure 3.** Schematic evolution of a perturbed black soliton in the case of an antisymmetric initial perturbation.

which may be rewritten now as $\Omega < \omega$, is consistent with the exact results by Zhao and Bourkoff. The scattering problem (4) with potential (77) can be solved exactly for arbitrary $\Omega$ and $\omega$. The solution is presented in appendix D.

## 6. On the creation of new solitons: general remarks

The criterion obtained in the previous section has a specific form. Now we derive a general relation allowing remarks about the soliton number to be made. To this end we rewrite the presentation (16) in the form

$$a(\lambda) = \exp\left(\frac{i}{2}\theta\right) \prod_{n=1}^{N} \frac{z(\lambda) - z_n}{z(\lambda) - \bar{z}_n} \exp(-iI_0 - iI(\lambda)) \tag{78}$$

where

$$I_0 = \frac{1}{2\pi} \int_{R_\omega} \frac{d\mu}{k(\mu)} \ln(1 + |b(\mu)|^2) \tag{79}$$

$$I(\lambda) = \frac{k(\lambda)}{2\pi} \int_{R_\omega} \frac{d\mu}{k(\mu)} \ln(1 + |b(\mu)|^2) \frac{1}{\mu - \lambda} \tag{80}$$

(here the unitarity of the scattering matrix (11) was used). Further analysis is based on the treatment of the integral $I(\lambda)$ near its singular points $\lambda = \pm\omega$, by analogy with the derivation of the so-called signum rule by Takhtajan and Faddeev (1986). Hereafter the variable $\lambda$ is taken from the upper half-plane: $\text{Im}\,\lambda > 0$.

It can be shown that when $\lambda$ is close to $\pm\omega$,

$$I(\lambda \approx \pm\omega) = \frac{i}{2} \ln\left(1 + \frac{b_\pm^2}{k^2(\lambda)}\right) \mp \frac{\pi}{2} + O(k(\lambda)) \tag{81}$$

where the logarithm branch is determined by the condition $\ln(1) = 0$. This gives

$$\exp(-iI(\lambda)) \sim \pm\left(1 + \frac{b_\pm^2}{k^2(\lambda)}\right)^{1/2} \qquad (\lambda \approx \pm\omega) \tag{82}$$

with the root branch defined by $(1)^{1/2} = 1$.

Noting that for $\text{Im}\,\lambda > 0$

$$\text{sgn Im}\left(1 + \frac{b_\pm^2}{k^2(\lambda \approx \pm\omega)}\right)^{1/2} = \mp 1 \tag{83}$$

while $\text{Im}\,k(\lambda)^{-1} < 0$ one can get

$$\left(1 + \frac{b_\pm^2}{k^2(\lambda)}\right)^{1/2} \sim \pm\frac{|b\pm|}{k(\lambda)} \qquad (\lambda \approx \pm\omega) \tag{84}$$

By using (78), (82) and (84) together with definition (32) one can show that

$$a_+ = i\tilde{a}|b_+| \tag{85}$$

$$a_- = i\tilde{a}|b_-|(-)^N \tag{86}$$

where the real constant $\tilde{a}$ is given by

$$\tilde{a} = \exp\left[i\left(\frac{\theta}{2} - I_0\right)\right] \prod_{n=1}^{N} \left(-\frac{z_n}{\omega}\right). \tag{87}$$

Multiplication of (85) and (86) leads to the relation

$$a_+a_- = (-)^{N+1}\tilde{a}^2|b_+b_-| \tag{88}$$

which gives the identity

$$(-)^N = -\mathrm{sgn}\; a_+a_- \tag{89}$$

where $N$ is the number of solitons and the 'residues' $a_\pm$ are defined by (32). The formula obtained is a universal one: it is valid for all initial conditions from the class (3).

Our further consideration is based on using formula (89) which may be rewritten in terms of $C(\lambda)$ (see (35)):

$$(-)^N = -\mathrm{sgn}\; C(-\omega)C(\omega). \tag{90}$$

In the case of weak initial fluctuations one can use the assumption that

$$\mathrm{sgn}\; C(-\omega)C(\omega) = \mathrm{sgn}\langle C(-\omega)C(\omega)\rangle \tag{91}$$

which is made on the grounds that for all $\lambda$, $\lambda'$

$$C(\lambda)C(\lambda') = \langle C(\lambda)C(\lambda')\rangle + \mathrm{O}((\delta q)^3). \tag{92}$$

First consider the case of a random phase modulation of the dark soliton (not only for the black one):

$$q(x, t=0) = q_s(x)\exp(\mathrm{i}\varphi(x)) \tag{93}$$

where $\varphi$ is a real random process defined by (41).

It can be shown after straightforward algebra that

$$\langle C(-\omega)C(\omega)\rangle_{\mathrm{ph}} = -\left(\frac{\pi}{2\nu}\right)^2 \int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{\Phi}(\xi)\xi^4 \,\mathrm{cosech}^2 \frac{\pi\xi}{2\nu} < 0. \tag{94}$$

Hence the one-soliton solution of the NSE under the boundary conditions (3) is 'unstable' under the phase modulation in the sense that any initial perturbation of the type (93) leads to the creation of the second soliton, at least.

Next, return to the case of the delta-correlated initial fluctuations (48)–(50) and (58) considered in section 4. Evaluation of the correlator from the RHS of (91) gives

$$\langle C(-\omega)C(\omega)\rangle_0 = \frac{8\omega^2\sigma_0^2}{5\nu}(4+\cos\theta - \cos 2\alpha + 6\cos\theta\cos 2\alpha). \tag{95}$$

By using this result, relation (91) may be written as

$$(-)^N = \mathrm{sgn}\left[\left(\gamma - \frac{\nu^2}{\omega^2}\right)(\gamma - \cos^2\alpha) - \gamma^2\right] \tag{96}$$

with $\gamma = \frac{5}{12}$. The constant $\gamma$ is determined by how quickly $\delta q$ decreases at infinity. In particular, for a perturbation of the form

$$\delta q(x) = \left(\cosh\frac{\nu x}{2}\right)^{-m} \psi(x) \tag{97}$$

with $\psi$ given by (48), parameter $\gamma$ depends on $m$:

$$\gamma = \tfrac{1}{2}\left(1 - \frac{m(m+1)}{(m+2)(m+3)}\right) \tag{98}$$

($\gamma = \frac{5}{12}$ corresponds to $m=1$).

Relation (96) shows that the parity of $N$ depends on the 'polarization' of fluctuations. This dependence is illustrated by figure 4.

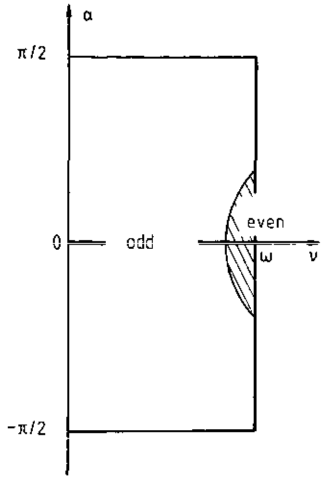


**Figure 4.** Dependence of the soliton number parity on the amplitude of the unperturbed soliton $\nu$ and 'polarization' of the delta-correlated fluctuations $\alpha$.

Considering the case of fluctuations of large correlation radius one can get from (B2) and (B3) together with (48)-(50) and (62) that

$$\langle C(-\omega)C(\omega)\rangle_\infty = (\sigma_\infty \cos\alpha)^2 w_+ w_- \tag{99}$$

where

$$w_\pm = \omega\, e^{-i\theta} \int_{-\infty}^{\infty} dx\, g(x) \left(\frac{e^{i\theta} \pm e^{\nu x}}{1+e^{\nu x}}\right)^2. \tag{100}$$

Evaluating the integral in (100) leads to the result

$$\operatorname{sgn}\langle C(-\omega)C(\omega)\rangle_\infty = \operatorname{sgn}\left(\frac{\lambda_s^2}{\omega^2} - \frac{1}{9}\right). \tag{101}$$

Hence the number of solitons is even if $|\lambda_s| < \frac{1}{3}\omega$.

## Conclusion

In conclusion we summarize the main points that distinguish the dynamics of a dark soliton from the dynamics of a bright one. These differences have as a beginning, from the mathematical point of view, two features of the scattering problem corresponding to the NSE with non-zero boundary conditions: the eigenvalues of the scattering problem are real, and the scattering data have, in general, singularities at $\pm\omega$.

Since the discrete spectrum is real, a dark soliton is one-parametrical, while a bright one is two-parametrical. That is why it was convenient to develop the perturbation theory for the variations $\delta\theta_l$ rather than for $\delta\lambda_l$, $\delta\nu_l$ directly. As a result a perturbation changes simultaneously both velocity and amplitude of a dark soliton, while in the bright soliton case the phase fluctuations do not result in fluctuations of amplitude and initial amplitude modulations do not affect velocity.

The main consequence of the fact that the scattering data are, in general, singular is the possibility of the creation of new solitons due to small perturbations. Though this effect is of not much importance from the point of view of applications (since amplitudes of 'newborn' solitons are as small as noise generated in a system), it is, in our opinion, very interesting in its own right and is the main feature distinguishing dark soliton dynamics from bright soliton dynamics.

It turns out that pure soliton solutions (i.e. solutions without a quasilinear constituent) are unstable: the number of solitons can be changed by a vanishingly small perturbation of many types. The physical origin of this phenomenon is the fact that creation of solitons is energetically profitable (energy of a dark soliton is less than the energy of the background alone). The mathematical explanation of this instability can be given as follows. The pure soliton case, when singularities are absent, is exceptional: $|a_s(\pm\omega)|=1$, $a_\pm=0$ (residues $a_\pm$ are defined by (32)). A perturbation, even if small, 'returns' us to the general case ($a_\pm\neq 0$). In this sense a quasilinear constituent may play a stabilizing role, since its presence implies $a_\pm\neq 0$. So, if the 'unperturbed' initial condition is not the pure soliton one ($a_\pm\neq 0$), a small initial perturbation leads to a small change of $a_\pm$: $a_\pm\to a_\pm+\delta a_\pm$, which is not crucial provided that unperturbed values of $a_\pm$ are non-zero.

**Acknowledgment**

We wish to thank Dr Yu S Kivshar for useful discussions.

## Appendix A

One of the one-soliton Jost functions can be written as follows

$$T_-(x;\lambda)=\begin{pmatrix} \dfrac{\omega\varepsilon\bar{q}_0+z}{\omega\varepsilon+z} & -\mathrm{i}\dfrac{\omega}{z}\dfrac{\omega+z\varepsilon\bar{q}_0}{\omega+z\varepsilon} \\ \mathrm{i}\dfrac{\omega}{z}\dfrac{\omega+z\bar{\varepsilon}q_0}{\omega+z\bar{\varepsilon}} & \dfrac{\omega\bar{\varepsilon}q_0+z}{\omega\bar{\varepsilon}+z} \end{pmatrix}\exp\left(\frac{kx}{2\mathrm{i}}\sigma_3\right) \tag{A1}$$

where $q_0=q_s/p$, $k(\lambda)$ and $z(\lambda)$ are defined by (9) and (10), correspondingly. The other Jost function may be obtained from (A1) by using the relation

$$T_+(x;\lambda)=T_-(x;\lambda)T_s^{-1}(\lambda) \tag{A2}$$

where $T_s(\lambda)=\mathrm{diag}(a_s(\lambda),\bar{a}_s(\lambda))$ with $a_s(\lambda)$ given by (22).

## Appendix B

It follows from (30) that

$$\frac{\delta a}{a_s}=\frac{\lambda-k}{2k}\int_{-\infty}^{\infty}\mathrm{d}x(\delta\bar{q}T_-^{(22)}T_-^{(21)}-\delta q T_-^{(11)}T_-^{(12)}) \tag{B1}$$

where $T_-^{(ij)}$ is an element of matrix $T_-$ (which is in the $i$th row and $j$th column).

Equation (B1) together with (A1) and (21) gives an expression for $C(\lambda)$ defined by (35):

$$C(\lambda)=2\,\mathrm{Re}\,\bar{\varepsilon}\int_{-\infty}^{\infty}\mathrm{d}x\,\delta q(x)w(x;\lambda) \tag{B2}$$

where function $w$ is given by

$$w(x;\lambda)=\frac{\omega}{8\cosh^2(\nu x/2)}\left(\bar{\varepsilon}\,\mathrm{e}^{\nu x}+\varepsilon\,\mathrm{e}^{-\nu x}+2\frac{\omega^2-\lambda\lambda_s}{\omega(\lambda-\lambda_s)}\right). \tag{B3}$$

## Appendix C

Consider the integral in the RHS of (37),

$$G=\frac{2}{\pi\omega^2}\int_{R_\omega}\mathrm{d}\lambda\,k(\lambda)\ln\left(1+\frac{C^2(\lambda)}{k^2(\lambda)}\right). \tag{C1}$$

When $\lambda$ is not close to $\pm\omega$, the logarithm may be expanded as a Taylor series ($\ln(1+C^2k^{-2})=C^2k^{-2}+\ldots$). This expansion fails near the edges of the continuous spectrum since $k(\pm\omega)=0$. Nevertheless it can be shown that

$$G=\frac{2}{\pi\omega^2}\int_{R_\omega}\frac{\mathrm{d}\lambda}{k(\lambda)}C^2(\lambda)+\mathrm{O}((\delta q)^3). \tag{C2}$$

To prove the last relation consider the difference

$$\mathscr{D}=G-\frac{2}{\pi\omega^2}\int_{R_\omega}\frac{\mathrm{d}\lambda}{k(\lambda)}C^2(\lambda) \tag{C3}$$

which can be written as

$$\mathscr{D}=H(C(\omega))-H(C(-\omega))+\mathscr{D}_+-\mathscr{D}_- \tag{C4}$$

where

$$H(C)=\frac{2}{\pi\omega^2}\int_\omega^\infty\mathrm{d}\lambda\left[k(\lambda)\ln\left(1+\frac{C^2}{k^2(\lambda)}\right)-\frac{C^2}{k(\lambda)}\right] \tag{C5}$$

and

$$\mathscr{D}_\pm=\frac{2}{\pi\omega^2}\int_\omega^\infty\mathrm{d}\lambda\left[k(\lambda)\ln\left(1+\frac{C^2(\pm\lambda)-C^2(\pm\omega)}{k^2(\lambda)+C^2(\pm\omega)}\right)-\frac{C^2(\pm\lambda)-C^2(\pm\omega)}{k(\lambda)}\right]. \tag{C6}$$

It can be shown that for small $C$

$$H(C)=-\frac{2}{3}\left(\frac{C}{\omega}\right)^3+O(C^4). \tag{C7}$$

As for the terms $\mathscr{D}_\pm$ in (C4), they may be written as follows:

$$\mathscr{D}_\pm=-\frac{2C^2(\pm\omega)}{\pi\omega^2}\int_\omega^\infty\frac{\mathrm{d}\lambda}{k(\lambda)}\frac{C^2(\pm\lambda)-C^2(\pm\omega)}{k^2(\lambda)} \tag{C8}$$

where only the lowest-order terms are retained.

Noting that $C(\lambda)$ is a meromorphic (in terms of $\lambda$) function, which implies that

$$C(\lambda)=C(\pm\omega)\pm\frac{k^2(\lambda)}{2\omega}C'(\pm\omega)+\mathrm{O}(k^3) \tag{C9}$$

when $\lambda\approx\pm\omega$, one can conclude that the integral in (C8) converges and is of order $(\delta q)^4$ ($C(\lambda)$ is of order $\delta q$). This proves relation (C2).

## Appendix D

Consider matrix $F(x;\lambda)$ defined by

$$F(x;\lambda)=E^{-1}(x;\lambda)T_{-}(x;\lambda). \tag{D1}$$

This matrix solves the equation

$$F_x=k^{-1}(q_{s0}-p)\begin{pmatrix} i\omega & \lambda\exp(ikx) \\ \lambda\exp(-ikx) & -i\omega \end{pmatrix}F \tag{D2}$$

under condition $F(-\infty,\lambda)=\mathbb{1}$ ($2\times 2$ unit matrix). Equation (D2) can be transformed into a second order differential equation which turns out to be of the hypergeometric type in terms of the variable

$$\tau=\tfrac{1}{2}[1+\tanh(\tfrac{1}{2}\Omega x)]. \tag{D3}$$

So, one can get

$$F^{(11)}(x;\lambda)=F\left(\frac{\omega}{\Omega},-\frac{\omega}{\Omega};\frac{k(\lambda)}{i\Omega};\tau(x)\right). \tag{D4}$$

The Jost coefficient $a(\lambda)$ of the scattering problem with $q_{s0}$ as a potential can be written as

$$a(\lambda)=\frac{ik}{\lambda}\frac{\Gamma^2\left(\dfrac{k}{i\Omega}\right)}{\Gamma\left(\dfrac{k+i\omega}{i\Omega}\right)\Gamma\left(\dfrac{k-i\omega}{i\Omega}\right)}. \tag{D5}$$

The discrete spectrum of the problem is given by

$$\lambda_1=0 \tag{D6}$$

$$\lambda_{2n}=-\lambda_{2n+1}=\sqrt{\omega^2-\nu_n^2} \qquad (n=1,\ldots,N_0) \tag{D7}$$

where

$$\nu_n=\omega-n\Omega \tag{D8}$$

and $N_0$ is the largest integer satisfying the condition

$$N_0<\frac{\omega}{\Omega}. \tag{D9}$$

The number of eigenvalues, i.e. the number of solitons, is

$$N=2N_0+1. \tag{D10}$$

Results (D6)-(D10) have been stated by Zhao and Bourkoff (1989).

## References

Bass F G, Kivshar Yu S, Konotop V V and Puzenko S A 1987 *Proc. 8th Int. School on Coherent Optics part 2 (Bratislava)* p 341

—— 1988 *Opt. Commun.* **68** 385

Elgin J N 1985 *Phys. Lett.* **110A** 441

Elgin J N and Kaup D J 1982 *Opt. Commun.* **43** 233

Elyutin S E, Maimistov A and Manykin E 1988 *Phys. Lett.* **132A** 25

Gredeskul S A and Kivshar Yu S 1989 *Phys. Rev. Lett.* **62** 977

Gredeskul S A, Kivshar Yu S and Yanovskaya M V 1990 *Phys. Rev.* A **41** 3994

Halas N J, Krökel D and Grischkowsky D 1987 *Appl. Phys. Lett.* **50** 886

Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 171

Kandidov V P Shlyonov S A 1984 *Izv. Vuzov Radiofizika*, **27** 1158 (in Russian)

Kivshar Yu S 1990 *Phys. Rev.* A **42**

Konotop V V 1989 *Kvant-Elektron.* **16** 1032 (1989 *Sov. J. Quantum Electron.* **19** 669)

Krökel D, Halas N J, Giuliani G and Grischkowsky D 1988 *Phys. Rev. Lett.* **60** 29

Takhtajan L A and Faddeev L D 1986 *Hamiltonian Approach in Soliton Theory* (Moscow: Nauka) (in Russian)

Tomlinson W J 1988 *Phys. Stat. Sol.* **150** 851

Tomlinson W J, Hawkins R J, Weiner A M, Heritage J P and Thurston R N 1989a *J. Opt. Soc. Am.* B **6** 329

Tomlinson W J, Stolen R H, Hawkins R J and Weiner A M 1989b *Nonlinear Guided-Wave Phenomena: Phys. Applic.* **2** 132

Vysloukh V A, Ivanov I V and Cherednik A V 1987 *Izv. Vuzov, Radiofizika* **30** 980 (in Russian)

Weiner A M, Heritage J P, Hawkins R J, Thurston R N, Kirschner E M, Leaird D E and Tomlinson W J 1988 *Phys. Rev. Lett.* **61** 2445

Zakharov V E and Shabat A B 1973 *Zh. Eksp. Teor. Fiz.* **64** 1627 (*Sov. Phys. JETP* **37** 823)

Zhao W and Bourkoff E 1989 *Opt. Lett.* **14** 703